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TOPOS THEORY AND COMPLEX ANALYSIS

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Introduction

It is known that a topos can be equipped with an internal language which allows one to do mathematics in the topos in much the same way one does mathematics with sets, with the restriction of not using choice and excluded middle, laws in general non valid in a topos.

The questions which we are concerned with here, are the following:

1) What is the meaning (possible relevance for actual mathematics) of the mathematics in a topos with a natural number object when one specializes to Sh(X), the topos of sheaves over a topological space?*

2) What external mathematical difficulties are mirrored internally in the non-validity of the excluded middle?

Here we consider the particular field of complex analysis. We state and prove the Weierstrass division theorem for functions of one complex variable in an elementary topos with natural number object. We then interpret this theorem in the topos of sheaves over an open set of $\mathbb{C}^{n-1}$ and we find that the exact meaning of the above mentioned theorem is the ordinary Weierstrass division theorem in n variables.

From this example which gives a partial answer to our first question we get a reply to our second question. Namely, difficulties imposed by working with parameters are mirrored into the no longer classical but intuitionistic logic of the topos.

We wish to express our thanks to A. Douady who suggested to us to focus our attention on Weierstrass division theorem as an interesting example to consider to give an answer to our questions.

This research is part of the author's doctoral dissertation.

1. Complex analysis in an elementary topos with a natural number object

Here we develop complex analysis in an elementary topos with natural number object $\mathbb{N}$, using the internal logic of that topos. We suppose the reader familiar with

* This question is also considered in [3].

the use of that logic and the construction of the real number object $\mathbb{R}$ of Dedekind cuts in the topos. A reader non familiar with these things can refer himself to [3] and [4]. For this section we were inspired by Bishop's work [1] on complex analysis but our context is different, in that we cannot use countable choice. Here we just state the right definitions for the topos context and the main theorems, some of them obtained after slight changes in Bishop's proofs.

Notation. In the topos language, if $\varphi(x_1, \ldots x_n)$ is a formula interpretable in $X_1 \times \cdots \times X_n$ we denote by $|\varphi(x_1, \ldots x_n)|_{x_1, \ldots, x_n} \rightarrowtail X_1 \times \cdots \times X_n$ the subobject of $X_1 \times \cdots \times X_n$ determined by φ.

We say that φ is valid ($\models \varphi$) iff

$$|\varphi(x_1, \ldots x_n)|_{x_1 \ldots x_n} \rightarrowtail X_1 \times \cdots \times X_n = 1_{X_1 \times \cdots \times X_n}.$$

Our complex number object is $\mathbb{C} = \mathbb{R} \times \mathbb{R}$ equipped with the usual operations: $\mathbb{C}$ is then a complete metric space (i.e. closed under limits of Cauchy sequences).

$\mathbb{C}$ is a field in the sense that $\models \neg z \in \text{units}(\mathbb{C}) \to z = 0$.

$\mathbb{C}$ is also an apartness field i.e. a ring with an apartness relation $\#$ ($\models z \# 0$ iff $\models |z| > 0$) where $0 \# 1$ and $\models z \# 0 \leftrightarrow z \in \text{units}(\mathbb{C})$.

Definitions.

— $U \rightarrowtail \mathbb{C}$ is open iff $\models \forall x \in U \quad \exists r \in \mathbb{R}^+ \quad B(x, r) \subset U$ where $\mathbb{R}^+ = |r > 0|_r \rightarrowtail \mathbb{R}$ and $B(x, r) = ||z - x| < r|_z \to \mathbb{C}$.

— $A \rightarrowtail \mathbb{C}$ is totally bounded iff:

$$\models \forall \varepsilon > 0 \quad \exists n \in \mathbb{N} \quad \exists f \in A^{[n]} \quad \forall z \in A \quad \exists i \leqslant n (z \in B(f(i), \varepsilon))$$

where $[n] = |i \leqslant n|_i \rightarrowtail \mathbb{N}$.

— $\overline{B(a, r)} = ||z - a| \leqslant r|_z \rightarrowtail \mathbb{C}$.

— If U is open $\overline{B(a, r)}$ is well contained in U ($\overline{B(a, r)} \subset \subset U$) iff $\models \exists r' > r$ $(B(a, r') \subset U)$.

Definition. Let U be an open set of $\mathbb{C}$ and $f: U \to \mathbb{C}$. f is continuous iff

$$\models \forall a \in U \quad \forall r > 0 (\overline{B(a, r)} \subset \subset U \to \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall z \quad \forall z'$$
$$(z, z' \in \overline{B(a, r)} \wedge |z - z'| < \delta \to |f(z) - f(z')| < \varepsilon)).$$

Remarks.

— Continuity is defined as uniform continuity on each closed sphere $\overline{B(a, r)} \subset \subset U$.

— In the same way holomorphy is defined as uniform differentiability on each closed $\overline{B(a, r)} \subset \subset U$.

— We remark that if $\gamma: [a, b] \to \mathbb{R}$ is a differentiable path such that: $\models \exists \varepsilon > 0 \ \forall z \in \gamma([a, b])$ $(\overline{B(z, \varepsilon)} \subset \subset U)$ and $f: U \to \mathbb{C}$ is continuous, then f is uniformly continuous on $\gamma([a, b])$, so we can define:

$$\int_\gamma f\,\mathrm{d}z = \lim_{m\to\infty}\sum_{j=1}^{m} f\left(\gamma\left(a+\frac{b-a}{m}j\right)\right)\gamma'\left(a+\frac{b-a}{m}j\right)\frac{b-a}{m}.$$

This justifies why we choose this definition of continuity for we want to be able to use the tool of the integral.

Definition. $f: U \to \mathbb{C}$ is analytic iff $\int_\gamma f\,\mathrm{d}z = 0$ for any triangular path γ whose interior is included in U.

Theorem. *f is holomorphic iff it is analytic.*

Proof. Our proof that if f is holomorphic, then it is analytic is slightly different from Bishop's proof which uses countable choice. We replace that argument by using uniform continuity of f on the interior of any triangular path in U.

We can define the winding number of a path with respect to z_0 by

$$O(\gamma, z_0) = \frac{1}{2\pi\mathrm{i}}\int_\gamma (z - z_0)^{-1}\mathrm{d}z.$$

We can define the homotopy of two paths as in the classical case.

Theorem. (Cauchy integral formula). *Let f be a holomorphic function on an open set U. Then $\forall z_0 \in U$ and $\forall\gamma$ closed path in $U - \{z_0\}$ null-homotopic in U:*

$$O(\gamma, z_0)f(z_0) = \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{f(z)}{z - z_0}\mathrm{d}z.$$

Proof. In Bishop's work this proof uses the fact that $\{f(z) - f(z_0)\}/(z - z_0)$ is holomorphic. We use the same fact for a topos, but we have to prove it differently, using the special case of the Cauchy integral formula for a sphere, which comes from homotopy arguments.

Theorem. *If f is holomorphic, then f has derivatives of all order.*

Theorem. *Let f be holomorphic on $B(z_0, r)$. Then the series*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)(z - z_0)^n}{n!}$$

converges uniformly on each $\overline{B(z_0, r')}$ $r' < r$, and is the unique such power series converging to f.

Theorem. (Liouville theorem). *Let $f: \mathbb{C} \to \mathbb{C}$ be holomorphic and $M, \in \mathbb{R}^+$ such that*

$$\forall z \in \mathbb{C} \quad |f(z)| \leqslant M, \quad \text{then} \quad \forall z \in \mathbb{C} \quad f(z) = f(0).$$

2. Interpretation of continuous and holomorphic functions

In this section we interpret the notion of continuous function and that of holomorphic function in a topos of sheaves on a topological space. We consider several levels of correspondence: between the open sets of $\mathbb{C}_X$, the sheaf of germs of continuous functions from X to $\mathbb{C}$, and those of $X \times \mathbb{C}$, between continuous (holomorphic) functions on these corresponding open sets. Through these relations we will also analyse the definitions of continuity and holomorphy which seemed at first unnatural, but find their justification in the following work.

We will sometimes make proofs in the context of elementary open sets in order to have a simpler presentation helping the reader to see the idea of the proof. The reader could easily convince himself that, to get the proof in the general case, one only has technical details to add.

The complex number object $\mathbb{C}_X$ in Sh(X) is given by: $\mathbb{C}_X(V) = \{f : V \to \mathbb{C} \mid f$ continuous$\}$ with the usual operations. For all the section we suppose that the space X is locally compact.

2.1. Interpretation of open sets

We can first remark that if U is an open set in $\mathbb{C}$, we get U_X as an open set in $\mathbb{C}_X$ by:

$$U_X(V) = \{f : V \to U \mid f \text{ continuous}\}.$$

As defined, U_X corresponds to the open set $X \times U$ of X. We now construct the bijection between the open sets of $\mathbb{C}_X$ in Sh(X) and the open sets of $X \times \mathbb{C}$:

$$\begin{aligned} &\mathrm{Open}(\mathbb{C}_X) \to \mathrm{Open}(X \times \mathbb{C}) \\ &\qquad U \mapsto \bar{U} = \{(x, z) \mid z \in U_x\} \end{aligned}$$

(here when we write U_x we just mean the set of the values of the germs at the point X).

In the other direction:

$$\begin{aligned} &\mathrm{Open}(\mathbb{C}_X) \leftarrow \mathrm{Open}(X \times \mathbb{C}) \\ &\qquad \hat{U} \leftarrow\!\shortmid U \end{aligned}$$

$$\hat{U}(V) = \{f : V \to \mathbb{C} \mid f \text{ continuous and, for all } x \in V,\ (x, f(x)) \in U\}.$$

Theorem.

$$\mathrm{Open}(\mathbb{C}_X) \xrightarrow{\sim} \mathrm{Open}(X \times \mathbb{C})$$

and the bijection is through the functions just defined.

Proof. i) If U is an open set in $\mathbb{C}_X$, $\bar{U}$ is open in $X \times \mathbb{C}$. Let $(x, z) \in \bar{U}$. There exists

$V \in \mathrm{Open}(X)$ and $z' \in U(V)$ such that $z'(x) = z$

$$\models \exists \varepsilon > 0 \qquad B(z', \varepsilon) \subset U.$$

Let W be an open set in V on which ε is globally defined and K' a neighbourhood of x relatively compact in W. We set

$$\varepsilon' = \min_{x \in K'} \varepsilon(x), \qquad \text{then} \qquad (x, z) \in K' \times B(z, \varepsilon') \subset \bar{U}.$$

ii) If U is open in X, $\hat{U}$ is an open set of $\mathbb{C}_X$. Let $f \in \hat{U}(V)$. We look for ε such that $B(f, \varepsilon) \subset \hat{U}(V)$. For all $x \in V$, let $U_x \times V_x$ be an elementary neighbourhood of $(x, f(x))$ in U, where V_x is $B(f(x), \delta_x)$ for a δ_x. We choose U_x small enough for:

$$x' \in U_x \implies |f(x) - f(x')| < \tfrac{1}{2}\delta_x$$

then we define ε locally on each U_x by $\varepsilon = \delta_x$ and we get

$$B(f, \varepsilon)(U_x) \subset \hat{U}(U_x).$$

iii) $\bar{\hat{U}} = U$ in $X \times \mathbb{C}$. If $(x, z) \in U$ there is an elementary neighbourhood $U_x \times V$ in U. Let c_z be the constant function on U_x with value z. Then

$$c_z \in \hat{U}(U_x) \implies (x, z) \in \bar{\hat{U}}.$$

If $(x, z) \in \bar{\hat{U}}$, there is V neighbourhood of x and $f \in \hat{U}(V)$ such that $f(x) = z$

$$f \in \hat{U}(V) \implies \forall y \qquad (y, f(y)) \in U$$

$$\text{in particular} \qquad (x, f(x) = z) \in U.$$

iv) $\hat{\bar{U}} = U$ in $\mathbb{C}_X$.

$$f \in \hat{\bar{U}}(V) \iff \text{for all } x \in V, \qquad (x, f(x)) \in \bar{U} \iff f \in U(V).$$

2.2. *Interpretation of continuous functions*

We have a bijection between $\mathrm{Cont}_{\mathrm{Sh}(X)}(U, \mathbb{C}_X)$ and $\mathrm{Cont}(\bar{U}, \mathbb{C}) = \{f : \bar{U} \to \mathbb{C} \mid f$ continuous$\}$ through the morphisms:

$$\phi : \mathrm{Cont}_{\mathrm{Sh}(X)}(U, \mathbb{C}_X) \to \mathrm{Cont}(\bar{U}, \mathbb{C})$$
$$f \mapsto \phi(f) = \bar{f}$$

$\bar{f}(x, z) = f_{U_x}(c_z)(x)$, where U_x is chosen such that there exists $\varepsilon > 0$ with $U_x \times B(z, \varepsilon) \subset \bar{U}$ (f being a morphism of sheaves, we have that $\bar{f}$ is well defined).

$$\psi : \mathrm{Cont}(\bar{U}, \mathbb{C}) \to \mathrm{Cont}_{\mathrm{Sh}(X)}(U, \mathbb{C}_X)$$
$$f \mapsto \psi(f) = \hat{f}$$

where $\hat{f}_V(h)(x) = f(x, h(x))$ for all $x \in V$, for all $h \in U(V)$.

Remark. We have $\phi(\psi(f))=f$ for all $f\in \mathrm{Cont}\,(\bar{U},\mathbb{C})$,

$$\phi(\psi(f))(x,z)=\psi(f)_{U_x}(c_z)(x)=f(x,c_z(x))=f(x,z).$$

We prove that φ and ψ are bijections inverse one to the other through the following steps:

If $f:U\to\mathbb{C}_X$ is continuous and $g,g'\in U(V)$ then

$$g(x)=g'(x)\Longrightarrow f_V(g)(x)=f_V(g)(x').$$

With the previous property we get $\psi\circ\phi=1$.

If $f:U\to\mathbb{C}_X$ is continuous so is $\bar{f}$.

If $f:\bar{U}\to\mathbb{C}$ is continuous so is $\hat{f}$.

Proposition. *Let $f:U\to\mathbb{C}_X$ continuous and $g,g'\in U(V)$ such that $g(x_0)=g'(x_0)$ then $f_V(g)(x_0)=f_V(g')(x_0)$.*

Proof. Let $y=g(x_0)=g'(x_0)$ and $W\times B(y,\eta)$ a neighbourhood of (x_0,y) in $\bar{U}$ where $W\subset U$. Then $c_y\in U(W)$.

f being continuous is in particular pointwise continuous. So given $\varepsilon'\in\mathbb{R}^+$ and setting $\varepsilon=c_{\varepsilon'}\in\mathbb{R}^+_X(W)$

$$\models\exists\delta>0\qquad |h-c_y|<\delta\to|f(h)-f(c_y)|<\varepsilon.$$

δ is locally defined. Let W' be an open set where δ is globally defined and $x_0\in W'$. Let K' be a relatively compact open set in W', neighbourhood of x_0 and let $\delta'=\min_{x\in K'}(x)$. Let $W''\subset K'$ be an open set containing x_0 such that:

$$x\in W''\Longrightarrow |g(x)-y|<\min(\delta,\eta)\text{ and }|g'(x)-y|<\min(\delta,\eta)$$

then $|f_{W''}(g|_{W''})-f_{W''}(c_y|_{W''})|<\varepsilon|_{W''}$ and $|f_{W''}(g'|_{W''})-f_{W''}(c_y|_{W''})|<\varepsilon|_{W''}$. In particular:

$$|f_{W''}(g|_{W''})(x_0)-f_{W''}(c_y|_{W''})(x_0)|<\varepsilon(x_0)=\varepsilon'\quad\text{for all}\quad\varepsilon'>0$$

so $f_{W''}(g|_{W''})(x_0)=f_{W''}(c_y|_{W''})(x_0)$.

As f is a morphism of sheaves we get $f_V(g)(x_0)=f_V(c_y)(x_0)$. In the same way:

$$f_V(g')(x_0)=f_V(c_y)(x_0)$$

so $f_V(g)(x_0)=f_V(g')(x_0)$.

Proposition.

$$\psi\circ\phi=1.$$

Proof. Let $f:U\to\mathbb{C}_X$ continuous. Let $h\in U(V)$ and $x\in V$

$$\psi(\phi(f))_V(h)(x)=\phi(f)(x,h(x))=f_{U_x}(c_{h(x)})(x)$$
$$=f_{U_x}(h)(x)\qquad=f_V(h)(x).$$

Proposition. *If $f : U \to \mathbb{C}_X$ is continuous, then $\bar{f} : \bar{U} \to \mathbb{C}$ is continuous.*

Proof. It is evident that f is continuous in the first variable. Let $(x, a) \in \bar{U}$ and $\varepsilon > 0$. We look for $\delta > 0$ such that:

$$a' \in B(a, \delta) \to |\bar{f}(x, a') - \bar{f}(x, a)| < \varepsilon.$$

Let U_x be an open set of X such that $c_a \in U(U_x)$. Considering $\varepsilon' = c_\varepsilon \in \mathbb{R}_X^+(U_x)$ $\quad \exists \delta' > 0$

$$|h - c_a| < \delta' \to |f(h) - f(c_a)| < \varepsilon'.$$

Let V be an open set on which δ is globally defined ($x \in V \subset U_x$) and K' a relatively compact open set of V containing x.

Let $\delta = \min \delta'(y) > 0$ then

$$\begin{aligned} a' \in B(a, \delta) \Longrightarrow |f(x, a') - f(x, a)| &= |f_{U_x}(c_{a'})(x) - f_{U_x}(c_a)(x)| \\ &= |f_{K'}(c_{a'}|_{K'})(x) - f_{K'}(c_a|_{K'})(x)| \\ &< \varepsilon'|_{K'}(x) = \varepsilon \end{aligned}$$

for $|c_{a'}|_{K'} - c_a|_{K'}| < c_\delta \leq \delta'|_{K'}$.

Remark. For this proof we just use that f is pointwise continuous.

Proposition. *Let $f : \bar{U} \to \mathbb{C}$ continuous, then $\hat{f} : \hat{U} \to \mathbb{C}_X$ is continuous.*

Proof. Let $a \in U(V)$, $r \in \mathbb{R}_X^+(V)$, such that $\overline{B(a, r)} \subset\subset U$, $V \in \text{Open}(X)$. Let $\varepsilon > 0$. For $x_0 \in V$ there exists a neighbourhood V_0 of x_0 and a section $\eta \in \mathbb{R}_X^+(V_0)$ such that $\overline{B(a, r + \eta)}|_{V_0} \subset U|_{V_0}$. Let K'_{x_0} be a relatively compact neighbourhood of x_0 such that for all $y \in \overline{K'_{x_0}}$:

$$\overline{B(a(y), r(y))} \subset \overline{B(a(x_0), r(x_0) + \tfrac{1}{2}\eta(x_0))} = B. \quad B \text{ is compact.}$$

If $\varepsilon' = \min_{x \in K'_{x_0}} \varepsilon(x)$, for each (x_0, z) with $z \in B$ there exists $V_z \times B(z, \delta_z)$, neighbourhood of (x_0, z) such that:

$$(y, z') \in V_z \times B(z; \delta_z) \Longrightarrow |f(y, z') - f(x_0, z)| < \tfrac{1}{2}\varepsilon'.$$

B is compact so

$$B = \bigcup_{i=1}^{n} B(z_i, \delta_{z_i}).$$

If $V' = K'_{x_0} \cap V_{z_1} \cap \cdots \cap V_{z_n}$, let δ' be a Lebesgue constant for that open covering of B. Then for $x \in V'$ and z, z' such that $|z - z'| < \delta'$, there exists a $B(z_i, \delta_{z_i})$ which contains both z and z'. So $|f(x, z) - f(x, z')| < \delta'$. We set $\delta = c_{\delta'} \in \mathbb{R}_X^+(V')$. Then, for all $W \subset V'$ and all $h, h' \in B(a, r)(W)$

$$|h-h'|<\delta \Longrightarrow \text{ for all } x \in W \qquad |h(x)-h'(x)|<\delta'$$
$$\Longrightarrow |f(x,h(x))-f(x,h'(x))|<\varepsilon' \leq \varepsilon(x).$$

So $|\hat{f}_W(h)-\hat{f}_W(h')|<\varepsilon$.

2.3. Some properties of continuous functions in topoi of sheaves

Definition. In an elementary topos with natural number object, $f: U \to \mathbb{C}$ where U is open in $\mathbb{C}$ is said to be pointwise continuous iff:

$$\models \forall x \in U \qquad \forall \varepsilon>0 \qquad \exists \delta>0 \qquad \forall y(|y-x|<\delta \to |f(y)-f(x)|<\varepsilon).$$

Theorem. *In the topos* $\mathrm{Sh}(X)$ *where* X *is locally compact, any pointwise continuous function* $f: U \to \mathbb{C}$ *is continuous (i.e. pointwise continuity equals uniform continuity on every closed sphere).*

Proof. The proof is contained in the proof of the last section: We have shown that if f is pointwise continuous, $\bar{f}$ is continuous and $\hat{\bar{f}}=f$. As $\hat{\bar{f}}$ is continuous, so is f.

This theorem gives us the fact that the notion of continuity is the same as the classical notion in all topoi $\mathrm{Sh}(X)$, where X is locally compact, which are the topoi appearing in analysis.

2.4. Interpretation of holomorphic functions

We will now show that there is a bijection:

$$\mathrm{Hol}_{\mathrm{Sh}(X)}(U, \mathbb{C}_X) = H_2(\bar{U}, \mathbb{C})$$

where $H_2(\bar{U},\mathbb{C})=\{f:\bar{U}\to\mathbb{C} \mid f$ is continuous, f is holomorphic in the second variable$\}$. To simplify the proof we will do it in the case where $\bar{U}$ is $X \times V$, for V open in $\mathbb{C}$. The reader will convince himself easily that he can get the general proofs, just adding a few more technical details.

Proposition. *Let* $f: U_X \to \mathbb{C}_X$ *holomorphic, then* $\bar{f}: X \times U \to \mathbb{C}$ *is holomorphic in the second variable.*

Proof. Let $x_0 \in X$, $\varepsilon>0$. We look for δ such that

$$|z-z_0|<\delta \to |\bar{f}(x_0,z)-\bar{f}(x_0,z_0)-\bar{f}'(x_0,z_0)(z-z_0)|<\varepsilon\,|z-z_0|.$$

Let $\varepsilon'=c_\varepsilon \in \mathbb{R}^+_X(X)$. We have $c_{z_0} \in U_X(X)$. Then

$$\exists \delta'>0 \qquad (|h-c_{z_0}| \to |f(h)-f(c_{z_0})-f'(c_{z_0})(h-c_{z_0})|<\varepsilon'|h-c_{z_0}|.$$

Let V be an open set of X where δ is globally defined and K' a relatively compact open set of V containing x_0.

Let $\delta = \min_{x \in K'} \delta'(x)$. Then

$$|z - z_0| < \delta \to |c_z|_{K'} - c_{z_0}|_{K'}| < \delta'|_K$$

$$\to |f_{K'}(c_z|_{K'})(x_0) - f_{K'}(c_{z_0}|_{K'})(x_0)(c_z|_{K'}(x_0) - c_{z_0}|_{K'}(x_0))|$$

$$< \varepsilon\, |c_z|_{K'}(x_0) - c_{z_0}|_{K'}(x_0)|$$

$$\therefore \quad |\bar{f}(x, z) - \bar{f}(x, z_0) - \bar{f}'(x, z_0)(z - z_0)| < \varepsilon\, |z - z_0|.$$

Montel's theorem in complex analysis states that a family of holomorphic functions defined on an open set U of $\mathbb{C}$ and uniformly bounded on each compact of U is uniformly equiderivable on each compact of U. This fact is used in the following proposition.

Proposition. *Let $f : X \times U \to \mathbb{C}$ continuous, holomorphic in the second variable. Then $\hat{f} : U_X \to \mathbb{C}_X$ is holomorphic.*

Proof. The derivative f' of f is given by $\partial f/\partial z$. Let $\overline{B(a, r)} \subset\subset U$ and $\varepsilon \in \mathbb{R}_X^+$. We will define δ locally on each relatively compact open set of X. Let K' be such an open set. Then $K = \bigcup_{x \in \bar{K}'} B(a(x), r(x))$ is compact in $\mathbb{C}$. The family of $\{f(x, -)|_K\}_{x \in K'}$ is uniformly bounded by $\max_{x \in \bar{K}', z \in K} |f(x, z)|$. This family is then uniformly equiderivable on K. If $\varepsilon' = \min_{x \in K'} \varepsilon(x)$, there exists $\delta' > 0$ such that for all $x \in K'$,

$$|z - z'| < \delta' \implies |f(x, z) - f(x, z') - \frac{\partial f}{\partial z}(x, z')(z - z')| < \varepsilon' |z - z'|.$$

We set $\delta = c_{\delta'}$ on K'.

Then, for all $W \subset K'$ and for all $h, h' \in B(a, r)(W)$

$$|h - h'| < \delta \implies |\hat{f}_{K'}(h)(x) - \hat{f}_{K'}(h')(x) - \frac{\partial f_{K'}}{\partial z}(h')(x)(h(x) - h'(x))|$$

$$= |f(x, h(x)) - f(x, h'(x)) - \frac{\partial f}{\partial z}(x, h'(x))(h(x) - h'(x))|$$

$$< \varepsilon' |h(x) - h'(x)| \leq \varepsilon(x) |h(x) - h'(x)|.$$

Let us consider the case where X is an open set of $\mathbb{C}^{n-1}$. We know that $\mathrm{Hol}_{\mathrm{Sh}(X)}(U, \mathbb{C}_X)$ interprets as $\mathcal{H}_n(\bar{U}, \mathbb{C})$ where $\mathcal{H}_n(\bar{U}, \mathbb{C}) = \{f : \bar{U} \to \mathbb{C} \mid f$ continuous, f holom[illegible] in $z_n\}$. The question we are now considering is the following: is there a way to [illegible]sent in the topos the holomorphic functions from $\bar{U}$ to $\mathbb{C}$? For this purpose we come back to the theorem:

$$\mathrm{Cont}_{\mathrm{Sh}(X)}(U, \mathbb{C}_X) \xrightarrow{\sim} \mathrm{Cont}(\bar{U}, \mathbb{C}).$$

$$f \mapsto \bar{f}.$$

Looking at $\bar{f}$, we see that the continuity of $\bar{f}$ in the first variable comes from the fact that we are considering continuous sections and the continuity in the second variable from the continuity of f. Considering that global holomorphy is the same as holomorphy in each variable, we could imitate this procedure for holomorphy using in the topos the subobject of $\mathbb{C}_X$ given by:

$$H(V) = \{f : V \to \mathbb{C} \mid f \text{ holomorphic}\}.$$

Given U an open set of $\mathbb{C}_X$, we write H_U for $H \cap U$. We can now show that the subobject of $\mathrm{Hol}_{\mathrm{Sh}(X)}(U, \mathbb{C}_X)$ given by the functions f such that $f(H_U) \subset H$, which we denote by $\mathrm{Hol}_{\mathrm{Sh}(X)}(H_U, H)$ is in bijection with $H(\bar{U}, \mathbb{C})$, the set of holomorphic functions from $\bar{U}$ to $\mathbb{C}$. As we change the object $\mathbb{C}_X$ for H, we must see what is complex analysis on H, and, supposing this is analogous to that on $\mathbb{C}$, for what reasons? This is the purpose of the next section.

Theorem. *There is a bijection* $\mathrm{Hol}_{\mathrm{Sh}(X)}(H_U, H) = H(\bar{U}, \mathbb{C})$ *where X is an open set of* $\mathbb{C}^{n-1}$.

Proof. If $f : \bar{U} \to \mathbb{C}$ is holomorphic, then for all $V \in \mathrm{Open}(X)$, for all $h : V \to U$ holomorphic $\hat{f}(h) : V \to \mathbb{C}$, $\hat{f}(h)(x) = f(x, h(x))$ is holomorphic.

Conversely, let $f : H_U \to H$ be holomorphic. To prove $\bar{f}$ holomorphic it is enough to prove that $\bar{f}$ is holomorphic in the first $(n-1)$ variables, for we already know that $\bar{f}$ is holomorphic in z_n. Let $(\mathbf{x}, z_n) \in \bar{U}$; there exists U_x neighbourhood of $\mathbf{x}$, such that $c_{z_n} \in H_U(U_x)$. There $\bar{f}(\mathbf{x}, z_n) = f_{U_x}(c_{z_n})(\mathbf{x})$ but $f_{U_x}(c_{z_n}) \in H(U_x)$. So $\bar{f}(-, z_n)$ is locally holomorphic in $\mathbf{x}$ for all z. This is enough to show that $\bar{f}$ is holomorphic in $\bar{U}$. As in the case of continuous functions we have proved the following theorem:

Theorem. *In a topos* $\mathrm{Sh}(X)$ *where X is locally compact, $f : U \to \mathbb{C}_X$ is holomorphic iff f is pointwise differentiable on U (where $f : U \to \mathbb{C}_X$ is pointwise differentiable with derivative g iff:*

$$\models \forall z \in U \quad \forall \varepsilon > 0 \quad \exists \delta > 0$$
$$\forall z'(|z - z'| < \delta \to |f(z') - f(z) - g(z)(z' - z)| < \varepsilon\, |z - z'|).$$

2.5. Objects for complex analysis

Here we try to list properties that a subobject A of $\mathbb{C}$ should have to be a suitable object for complex analysis in the topos: we would then take the trace of the open sets of $\mathbb{C}$ on A and restrain variables to A. For that purpose we remind ourselves the complex analysis developed in Section 1 and we see that A must verify the following properties:

1) A is a subfield of $\mathbb{C}$ for the two definitions of field:

$$\models \neg\, z \in \mathrm{Unit}(A) \to z = 0$$

2) A is Cauchy-complete.

3) A is dense in $\mathbb{C}$ in the following sense: $A \supset \mathbb{Q} \times \mathbb{Q}$. (*Remark*: f is continuous on $\mathbb{C}$ iff f is continuous on $\mathbb{Q} \times \mathbb{Q}$ iff f is continuous on A. Same thing for f holomorphic.)

4) If $f : U \cap A \to A$ is holomorphic, the power series in a neighbourhood of $z \in U \cap A$ has coefficients in A and conversely.

5) We must restrain ourselves to paths γ such that if $f : U \cap A \to A$ is holomorphic and $\gamma \subset \subset U$ then $\int_\gamma f \, dz \in A$. Choice of paths for complex analysis in A:

If $\gamma : [a, b] \to \mathbb{C}$, we can replace γ by $\tilde{\gamma} : [0, 1] \to \mathbb{C}$, where $\tilde{\gamma}(t) = \gamma(a + t(b - a))$. In our context it will be enough to consider paths with the property:

$$\gamma([0,1]_c) \subseteq A \qquad [0,1]_c = [0,1] \cap \mathbb{R}_c$$

where $\mathbb{R}_c$ is the Cauchy Real object. In fact

$$\sum_{i=1}^{n} f(\gamma(t_i))(\gamma(t_i) - \gamma(t_{i-1})) \in A \qquad \text{where } t_i = i/n$$

so the limit $\int_\gamma f \, dz \in A$. We can easily verify that the subobject H of $\mathbb{C}$ defined in the last section has all these properties.

Remark. Property 4) is a consequence of properties 1)–3) under restriction 5).

3. Analysis with parameters and analysis in topoi

In this section we want to make the connection between the two preceding chapters, showing how certain complex analysis theorems in the topos could be interpreted and conversely how we could look for a topos formulation for some classical theorems. Considering a holomorphic function $f : X \times U \to \mathbb{C}$ and its corresponding function $\hat{f} : H_U \to H$ in $\mathrm{Sh}(X)$ we see that the variable in U is the only one we can speak of in $\mathrm{Sh}(X)$. This fact suggests we consider the variables in X as parameters. We compare two contexts: the context of the language in the topos which obeys an intuitionistic logic and the context of a language with parameters. Here we specially study the Weierstrass division theorem which is a typical theorem parameters and a fundamental theorem in analysis. We look here for a theorem in a topos that could be interpreted as that theorem.

3.1. *Interpretation of the integral*

We saw that we could interpret functions in $\mathrm{Sh}(X)$ as functions of several variables or as functions depending upon parameters. The easiest way to interpret a path in $\mathbb{C}_X$ is to think of it as a path depending upon parameters. Then if γ is a path in U_X and $f : U_X \to \mathbb{C}_X$ is holomorphic

$$\left(\int_{z\in\gamma} f(z)\,\mathrm{d}z\right)(x)=\int_{z\in\gamma_x}\bar{f}(x,z)\,\mathrm{d}z.$$

As an immediate consequence we see that the existence of the integral interprets as the following theorem.

Theorem. *Let $f_x : U \to \mathbb{C}$, depending continuously (holomorphically) upon parameters in X, and let γ_x be a piecewise differentiable path in U depending continuously (holomorphically) upon $x \in X$. Then $\int_{z\in\gamma_x} f_x(z)\,\mathrm{d}z$ depends continuously (holomorphically) upon $x \in X$.*

3.2. Weierstrass division theorem in the topos

We first recall the classical Weierstrass division theorem for which we will try to find a formulation in the topos.

Theorem. *Let $f : U \to \mathbb{C}$ be holomorphic, U neighbourhood of 0 in $\mathbb{C}^n$. If f is regular of order p in z_n at 0*

$$\left(i.e.\quad \frac{\partial^p f}{\partial z_n^p}(0)\neq 0 \quad and \quad \frac{\partial^i f}{\partial z_n^i}(0)=0, \quad for\ i<p\right)$$

then, if $g : U \to \mathbb{C}$ is holomorphic, there exists $W \subset U$ neighbourhood of 0 in $\mathbb{C}$, there is a unique $h : W \to \mathbb{C}$ holomorphic and a unique polynomial P in z_n of degree less than p with coefficients holomorphic in $(z_1 \cdots z_{n-1}) \in \Pi_{-n}(W)$ such that $g = fh + P$.

In this theorem $z_1, \ldots z_{n-1}$ are parameters. We look in the topos for a one variable theorem that will interpret in $\mathrm{Sh}(U)$ where U is an open set in $\mathbb{C}^{n-1}$.

$f : U \to \mathbb{C}$ gives $\hat{f} : H_{U_X} \to H$ holomorphic in the topos. The next problem is to formulate the hypothesis on f. We see that $\hat{f}^{(p)}(0) = (\partial^p f/\partial z_n^p)(-,0) : X \to \mathbb{C}$; but $(\partial^p f/\partial z_n^p)(0) \neq 0$. So in a neighbourhood of 0 in $\mathbb{C}^{n-1}$, we have $(\partial^p f/\partial z_n^p)(x,0) \neq 0$. We can suppose that X is that neighbourhood and we have there: $\hat{f}^{(p)}(0) \# 0$. There is a problem with the rest of the hypothesis: $\partial^i f/\partial z_n^i(0) = 0$, for this is not a local property: so there is no formula in the topos language to express this hypothesis. We see that if we want a theorem in a topos that could reflect some of the Weierstrass division theorem, as we have a weaker hypothesis, that theorem in the topos must have a weaker conclusion. From what we know about the language and its interpretation it seems reasonable to prove the following theorem whose formulation was first given by Michael Fourman.

Theorem (Weierstrass division theorem in the topos). *Let $f : U \to \mathbb{C}$ holomorphic, U neighbourhood of 0 be such that $f^{(p)}(0) \# 0$. Then $\forall g : U \to \mathbb{C}$ holomorphic; either $\exists i < p\ f^{(i)}(0) \# 0$ or $\exists W \subset U$ neighbourhood of 0*

$\exists! h : W \to \mathbb{C}$ *holomorphic*

$\exists! P$ *polynomial of degree less than p such that $g = fh + P$.*

It is easily seen that, restricted to H and interpreted on an open set of $\mathbb{C}^{n-1}$, this theorem gives the classical Weierstrass division theorem.

Proof of the theorem in the topos.

— Let $A(z)=\sum_{n\geq 0} a_n z^n$. We define $a(A)(z)=\sum_{n\geq 0}^{p-1} a_n z^n$ and $q(A)(z)=\sum_{n\geq p} a_n z^{n-p}$.

Then $A(z)=a(A)(z)+z^p q(A)(z)$.

— $q(f)(0)=(1/p!)f^{(p)}(0) \# 0$.

Then $\exists U_1 \subset U$ neighbourhood of 0 such that $\forall z\in U_1$ $|q(f)(z)|>\frac{1}{2}|q(f)(0)|$.

— To look for h such that $g-hf$ is a polynomial of degree $\leq p-1$ is the same as to look for h such that $q(g)=q(hf)$. As $q(f)$ is invertible on U_1, we will rather look for $k=hq(f)$. We have: $hf=ha(f)+z^p hq(f)$ so

$$q(g)=q(hf) \qquad q(g)=q(ha(f))+hq(f)$$
$$=-q(lk)+k, \quad \text{where} \quad l=-a(f)/q(f).$$

— We define the following linear operation on power series:

$$m(A)=q(lA).$$

We set $G=q(g)$, then we look for k such that $k=G+m(k)$. When iterating we get: $k=G+m(G)+\cdots m^q(G)+m^{q+1}(k)$ $\forall q$. This suggests we define $k=G+m(G)+m^2(G)+\cdots$. The problem is to show that this series converges.

— Let $l=\sum_{n\geq 0} b_n z^n$ in $\overline{B(0,\rho)}\subset U_1$ and $\nu=\sum_{n\geq 0}|b_n|\rho^n$ we have $\models \nu \# 0 \vee \nu<\rho^p/2^p$. In the first case:

$$\nu \# 0 \rightarrow \exists N \qquad \sum_{n\leq N}|b_n|\rho^n \# 0$$
$$\rightarrow \exists i\leq N \qquad b_i \# 0.$$

If $f=\sum_{n\geq 0} a_n z^n$ and $q(f)^{-1}=\sum_{n\geq 0} d_n z^n$, as $l=a(f)q(f)^{-1}$, we have $l_q=\sum_{n=0}^{\min(p-1,i)} a_n d_{i-n}$ so

$$b_i \# 0 \rightarrow \exists j\leq \min(p-1,i) \qquad a_j d_{i-j} \# 0$$
$$\rightarrow \exists j<p \qquad a_j \# 0 \rightarrow \exists j<p \qquad f^{(j)}(0) \# 0.$$

We now continue the proof in the second case: In $\overline{B(0,\rho)}$ we set $G(z)=\sum_{n\geq 0}\alpha_n z^n$.

Let $\mu=\sum_{n\geq 0}|\alpha_n|\rho^n$, then G has $G'=\mu/(1-z/\rho)$ as a majorant series, and l has $l'=\nu/(1-z/\rho)$.

— We set $m'(A)=q(l'A)$. If A has majorant A', then $m(A)$ has majorant $m'(A')$. To show that k converges it is enough to show that $k'=G'+m'(G')+\cdots$ converges. For that purpose we build a series $\varphi(z)$

majorant of $1/(1-z)$ with the property that: $m'(\varphi(z)) = 2^{p+1}\nu\varphi(z)$. Let $\varphi(z) = \sum_{n\geqslant 0} c_n z^n$; we define

$$\begin{cases} c_i = 2^i & \text{for } i < p \\ c_p = 2^p + 1 & \\ c_{n+p+1} = 2^{p+1}(c_{n+1} - c_n) & \text{for } \quad n \in \mathbb{N}; \end{cases}$$

as $c_{n+p+1} = 2^{p+1}(n - \sum_{i\leqslant n+p} c_i)$, we see immediately that

$$m'(\varphi(z)) = 2^{p+1}\nu\varphi(z).$$

We now check that $\varphi(z)$ is a majorant of $1/(1-z)$, i.e. $\forall i \quad c_i \geqslant 1$. But $c_0 \geqslant 1$, so it is enough to show that $\forall n \quad c_n < c_{n+1}$

— it is true for $i \leqslant p$.

— We suppose that it is true $\forall n' \leqslant n+p$. Then $c_{n+p+1} - c_{n+p} = 2^{p+1}(c_{n+1} - c_n) > 0$. We now look for an open set on which φ converges. For that we show by induction that $\forall n \quad c_n \leqslant 2^{n+1}$

— it is true $\forall n \leqslant p$

— if it is true $\forall m \leqslant n+p$ then $c_{n+p+1} = 2^{p+1}(c_n - c_{n-1}) \leqslant 2^{p+1}c_n \leqslant 2^{p+1}2^{n+1} = 2^{n+p+2}$, so φ converges in $B(0, \frac{1}{2})$ and $\varphi(z/\rho)$ converges in $B(0, \rho/2)$.

— Then G has majorant $G_1' = \mu\varphi(z/\rho)$ and $k_1' = G_1' + m'(G_1') + \cdots$ is a majorant of k. k_1' has majorant:

$$\mu\left(\varphi\left(\frac{z}{\rho}\right) + \nu\varphi\left(\frac{z}{\rho}\right)\frac{2^{p+1}}{\rho^p} + \cdots \nu^s\varphi\left(\frac{z}{\rho}\right)\frac{2^{s(p+1)}}{\rho^{ps}} + \cdots\right).$$

So k converges in $W = B(0, \rho/2)$.

— Unicity: Let h and h' be two solutions and $k' = h'q(f)$; we must have $m(k - k') = k - k'$. We can suppose that h' is defined on $W' = B(0, \rho')$ $\rho' \leqslant \rho$. So the problem is to show that if $m(A) = A$ then $A = 0$. In fact $A = m(A) = m^2(A) = \cdots$. If A has majorant $\kappa/(1 - z/\rho')$ which itself has majorant $\kappa\varphi(z/\rho')$ then $m^s(A)$ has majorant $\kappa\nu^s\varphi(z/\rho')2^{s(p+1)}/\rho'^{sp}$. But $\nu \# 0 \vee \nu < 2^{p+1}/\rho'^p$.

— If $\nu \# 0$ we conclude that $\exists i < p$ such that $f^{(i)}(0) \# 0$.

— If $\nu < 2^{p+1}/\rho'^p$ then $|m^s(A)| \to 0$, so $A = 0$.

The unicity of $P = g - hf$ comes from that of h.

As a consequence of the theorem we have the theorem of approximation of zeros which interprets exactly as the implicit function theorem.

Theorem. *Let $f: U \to \mathbb{C}$ be holomorphic, then*

$$\models f'(0) \# 0 \to f(0) \# 0 \vee \exists\delta > 0 \quad \exists!\, y(|y| < \delta \wedge f(y) = 0).$$

In conclusion we can say that validity in topoi gives continuity in parameters or holomorphy in parameters depending on which object we interpret. The statement of the Weierstrass division theorem in the topos interprets as a formula in a language with parameters. In such a language we have to deal again with the difficulty that we can just partially express the fact that a function is regular of order p in z_n at a point. In proving the division theorem in one variable in the topos we used with some modifications the proof of the theorem for power series from [5]. Comparing the proof in the topos with the proof with parameters [5] we see that the difficulties are the same, and since any proof in the topos interprets as a proof with parameters, the proof in the topos is at least as complex as the classical one.

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